J. mar. biol. Ass. India, 1972, 14 (1): 84-101

ON THE MATHEMATICAL DESCRIPTION OF THE LOWER TROPOSPHERIC DRIFT FLOW OVER THE INDIAN OCEAN DURING THE SE/SW MONSOON*

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ABSTRACT

During the SE/SW monsoon, the dominant quasi-stationary flow pattern of the lower troposphere over much of the Indian Ocean is characterised by a flow pattern often referred to as a DRIFT flow.

Mathematical description of the flow using particle dynamics was first given by Gordon (1953) and later by Johnson and Mörth (1959). Although the pattern of their solutions resemble the actual flow, yet computed wind speeds give unrealistic estimates with the consequence that their analytic results could not be used for estimating derived quantities such as divergence and vorticity.

This paper gives a mathematical description which is partly based on empirical results pertaining to the flow. The description is free from the defects of Gordon, and of Johnson and Mörth models.

INTRODUCTION

For a long time, the low level flow of the atmosphere has received considerable attention. Throughout the entire globe the seasonal changes of the weather have been always accompanied by changes in the large scale wind field both at the upper levels and in the lower troposphere. The most widely known of these changes in the tropical zones are the monsoon circulations of the lower troposphere.

The problem of the circulation of the lower troposphere has received considerable attention, to such an extent that the theme of a symposium held in India in early 1958 was centred around the topic. Similarly in other symposia on tropical meteorology, such circulations have been given considerable attention.

Although monsoonal circulations are not restricted to the Indian Ocean alone, this paper will be primarily concerned with the Indian Ocean southeast/southwest (SE/SW) monsoon, which is at its peak development during July/August. The treatment here is readily extended to other areas of the tropics. Such should be the case for the SE/SW monsoon over the Atlantic Ocean affecting West Africa during July/August.

DESCRIPTION OF THE SE/SW MONSOON FLOW

During the Northern Hemisphere summer, the low level pressure field is such that the subtropical anticyclone is well developed over the Malagasy island and the

* Presented at the 'Symposium on Indian Ocean and Adjacent Seas—Their Origin, Science and Resources' held by the Marine Biological Association of India at Cochin from January 12 to 18, 1971.

neighbouring oceanic region. The central surface pressure in this anticyclone varies between 1022 to 1025 mbs. The mean central latitude of this anticyclone is at about 30°S. The pressure gradually decreases northwards culminating at the equatorial trough whose axis extends from the Sahara at about latitude 20°N eastwards through Arabia ; then northeastwards towards the Indian subcontinent. The central pressure of this mean low varies from 995-1000 mb.

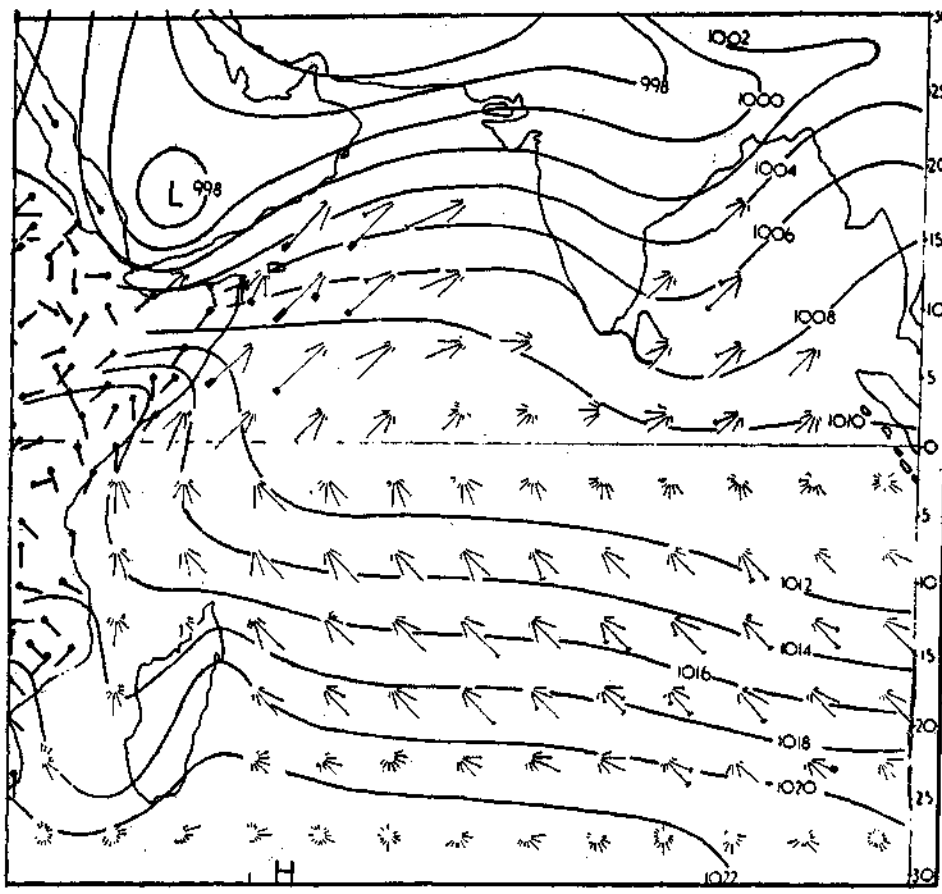

Fig. 1. Mean surface pressure and wind roses over East Africa and the adjacent Indian Ocean in July (After Thompson, 1965).

Very near the equator there are on occasions low values in the pressure field. Usually the magnitude is of the order of 1010 mbs. Fig. 1 is a mean surface pressure situation over East Africa and the adjacent Indian Ocean in July. Fig. 2 shows the contour at the 850 mb level a height of about 5000 ft above mean sea level. Essentially this pattern is similar to the pattern at the surface.

Fig. 3 is the July mean streamline field at the surface. The field depicts the monsoonal flow from the Southern Hemisphere Subtropical Anticyclone to the

Northern Hemisphere equatorial trough. Fig. 4 shows the situation at the 850 mb level.

If one observes the July daily analyses of the streamline fields from the surface up to a level below the 850 mb, one finds that the patterns do not change appreciably from day to day. Further, the patterns bear remarkable resemblance to the mean July field.

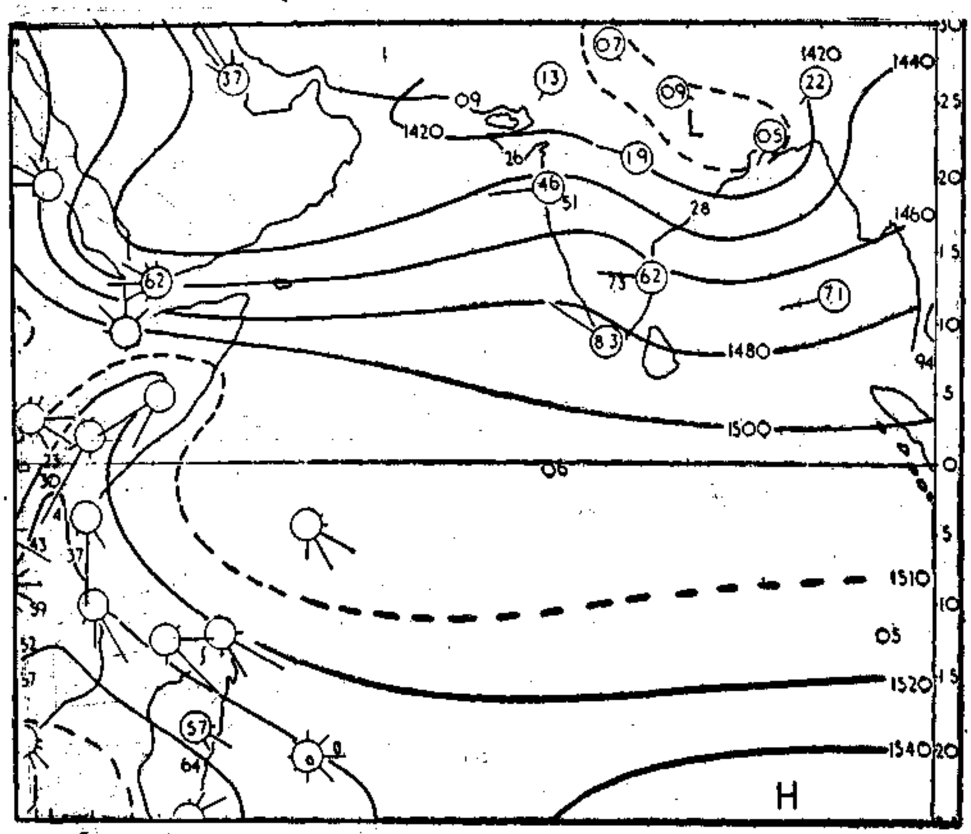

Fig. 2. Mean contour field at the 850 mb level in July (After Thompson, 1965).

Exception to this situation is the occasion when disturbances in the form of easterly waves perturb the flow in Indian Ocean moving eastwards across the northern tip of Malagasy to the East African coast. An example of such Easterly perturbation has been described by Fremming (1970) and an example of the disturbance is shown in Fig. 5. Although the figure is for September, similar cases are observed in July/August. Occasionally some disturbances originate far east of the East African coasts over the Indian Ocean and the first indication of their existence is when they bring rain over the East African coastal stations. The three dimensional structure of such disturbances are poorly understood because of paucity of upper air data. Cold fronts penetrating into low latitudes from the longitudes of the Mozambique channels are further examples of disturbances over the Indian Ocean which perturb the otherwise steady monsoonal flow.

But although these perturbations do disturb the monsoon flow, yet the constancy of the flow is high. The wind speeds themselves are not properly known because of lack of adequate observational data over much of the oceanic region. However at the lower levels Jenkinson (1968) in an unpublished manuscript gave a rough

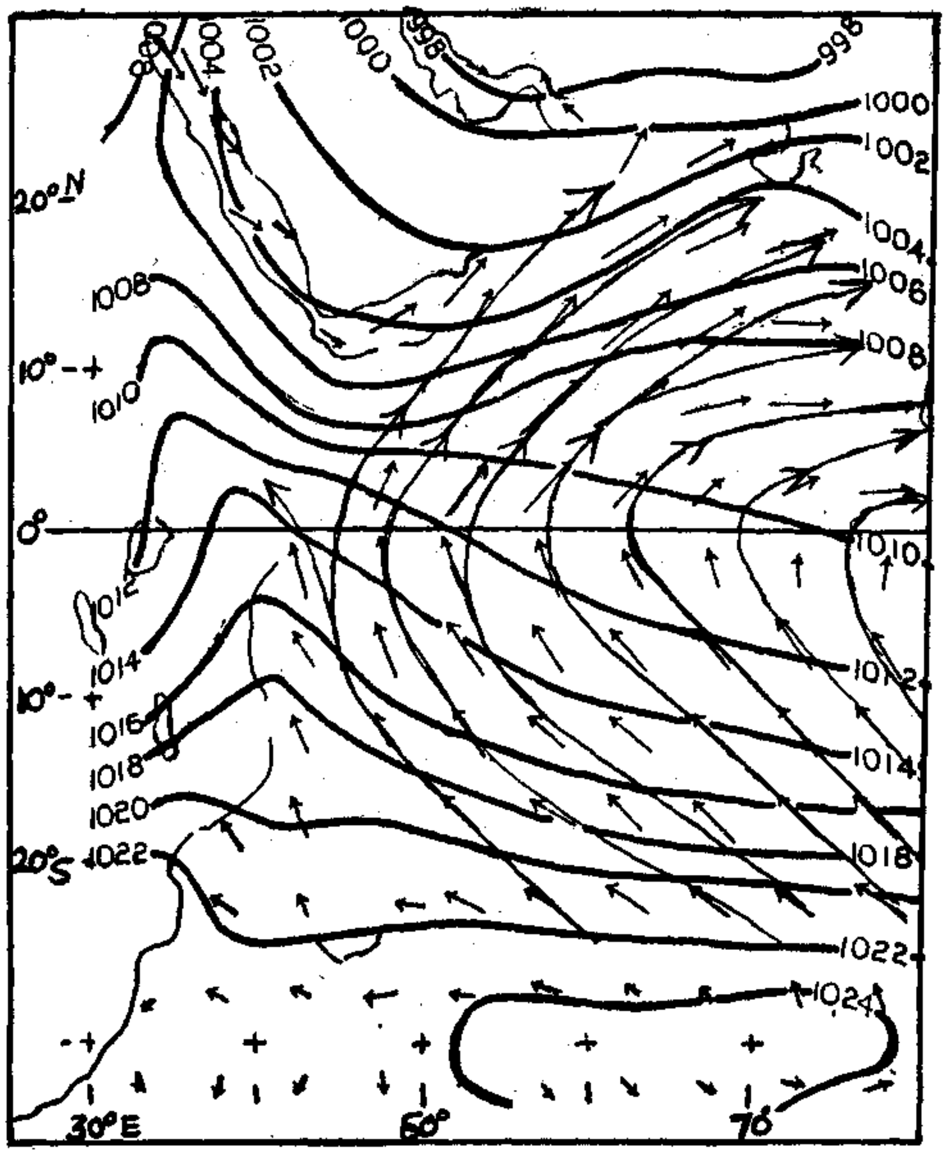

Fig. 3. Mean streamline at the surface in July.

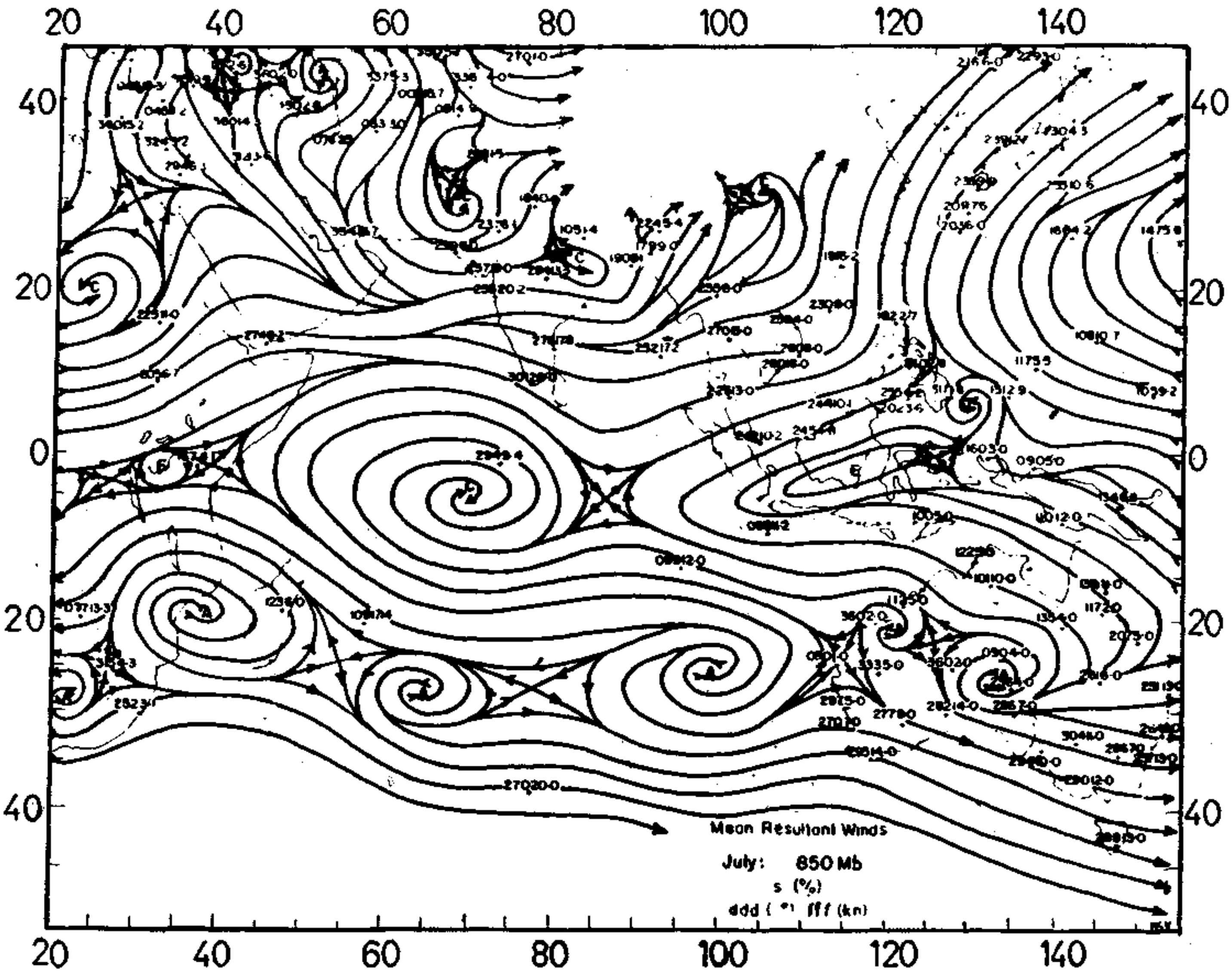

Fig. 4. Mean streamline at the 850 mb level in July, after Raman and Dixit (1963).

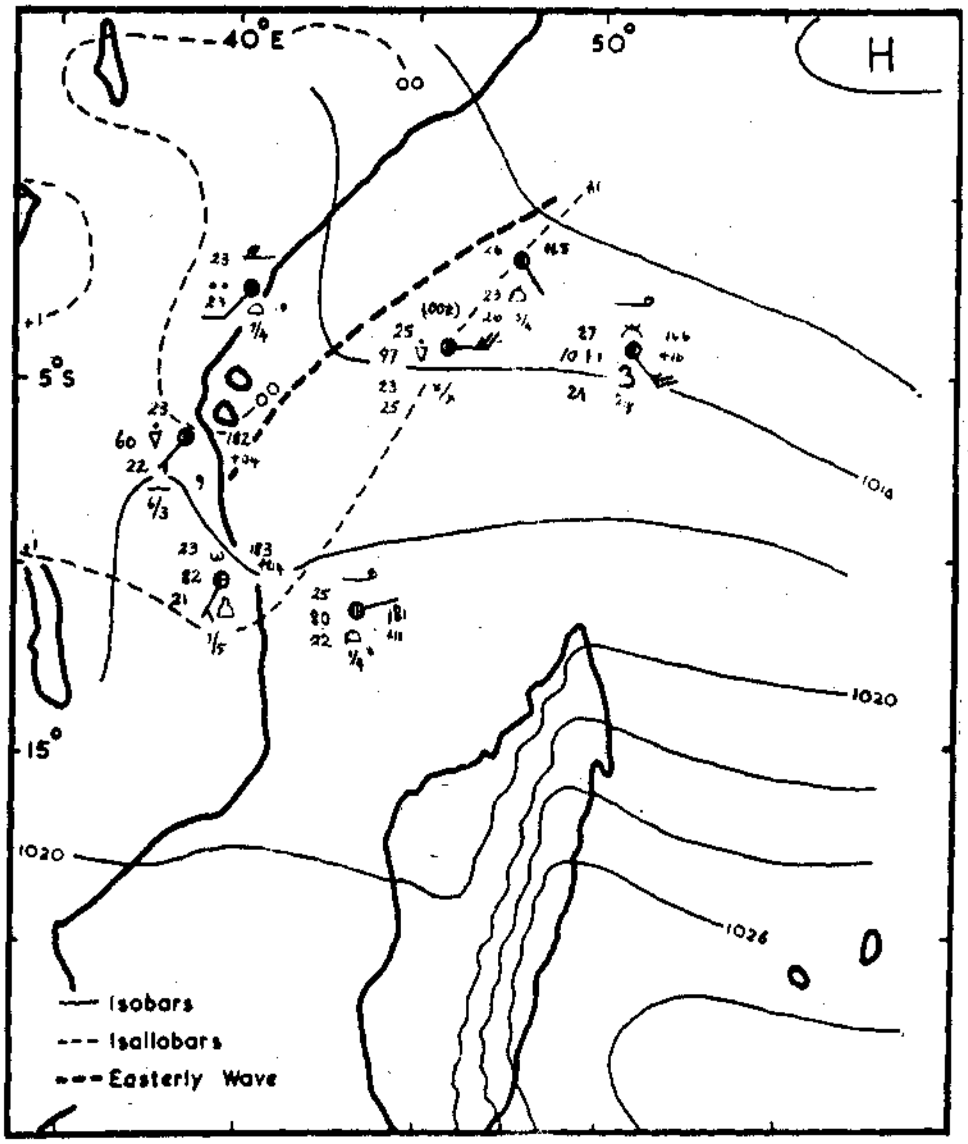

Fig. 5. An example of an easterly perturbation over the Indian Ocean. The figure is the 0600 Z surface chart for 5 September 1967, after Fremming (1970).

[Z]

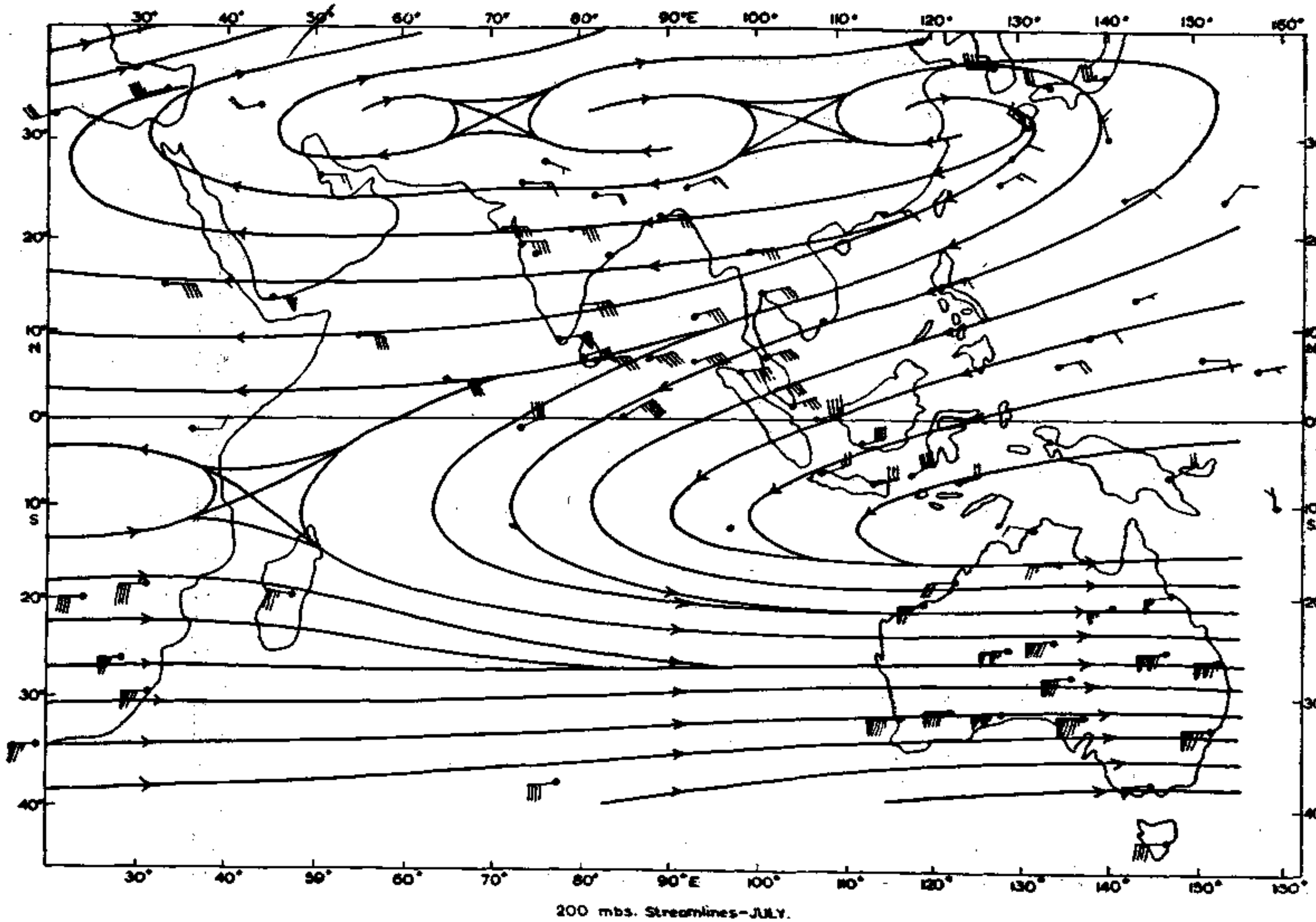

Fig. 6. Mean streamline at the 200 mb level in July showing upper level drift pattern, after Frost and Stephenson (1963).

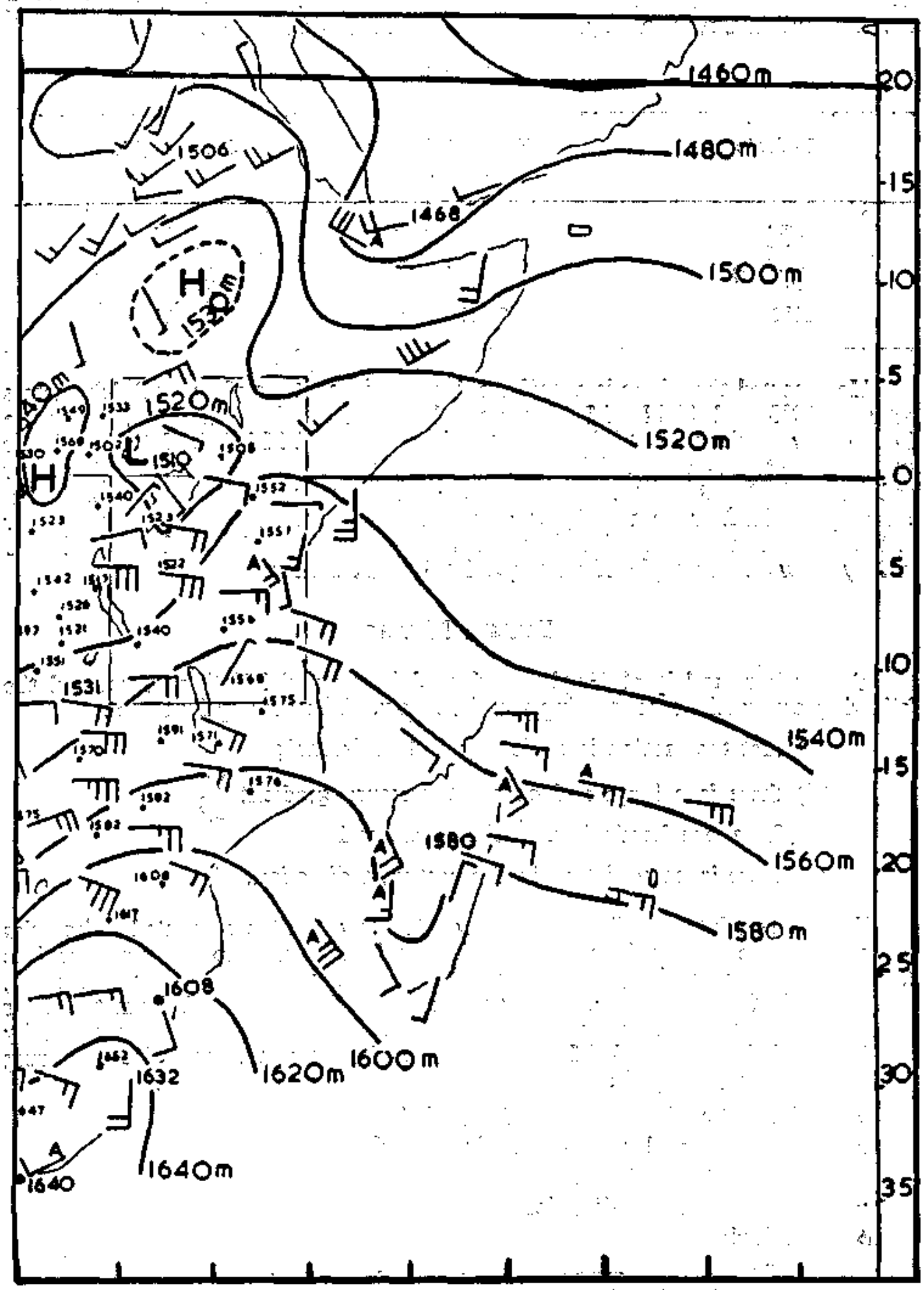

Fig. 6a. An example of the 850 mb drift flow on 9 August 1959 over East Africa and the adjacent Indian Ocean, after Johnson and Mörth (1959).

estimate of the actual wind for the low level (below the 850 mb level) SE/SW monsoon over the Indian Ocean in July. This is shown in Table 1 below.

TABLE 1

Latitude	Easterly/Westerly (knots)	Southerly/Northerly (knots)
10°N	20 Westerly	Nil
5°N	10 Westerly	10 Southerly
0 (Equator)	Nil	20 Southerly
5°S	10 Easterly	10 Southerly
10°S	20 Easterly	Nil

He reported that there is a uniform decrease in contour of about 9 dekameters between 10°S and 10°N. Mathematical descriptions of the flow using particle dynamics were given by Gordon (1953) and Johnson and Mörth (1959). The latter authors gave the name DRIFT to this type of flow from one hemisphere to the other and emphasized that the DRIFT is often seen at other levels of the atmosphere with cross equatorial flow from a higher contour in one hemisphere to a lower contour in the other hemisphere. Fig. 6 which is the mean streamline field at the 200 mb in July shows an example of an upper level DRIFT flow.

EARLIER MODELS

The earlier models of Gordon and of Johnson and Mörth incorporated the following assumptions concerning the drift flow. The assumptions are:

(i) that the motion is stationary and horizontal;

(ii) there is a large and constant pressure gradient $\frac{\delta p}{\delta y}$ from one hemisphere to the other; and

(iii) there is no variation in the east-west direction.

The primary interest of the authors was to show that the computed trajectories resembled those observed in quasi-stationary drift flow, although at a later date Kruger (1960) suggested that starting with the same assumptions as those of Gordon, Johnson and Mörth that the drift flow solutions could be obtained from the Euler equations. In what follows we shall derive the drift flow solutions as given by Gordon, and Johnson and Mörth using the Euler's equations. The following notation will be employed:

u is the eastward component of the wind
v is the northward component of the wind
x is the west-east direction
y is the south-north direction
ϕ is latitude
t is time
g is the acceleration due to gravity
$\mathfrak{z}$ is the height

P is the pressure

Ω is the angular velocity of the earth

The two horizontal equations of motion are :

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - 2\,\Omega\, v \sin\phi = g\frac{\partial \Phi}{\partial x} \tag{1}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + 2\,\Omega\, u \sin\phi = -g\frac{\partial \Phi}{\partial y} \tag{2}$$

For steady flow with no variation in the east-west direction

$$\frac{\partial u}{\partial t} = \frac{\partial v}{\partial t} = \frac{\partial \Phi}{\partial t} = 0 \text{ (steady state)} \tag{3}$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial x} = \frac{\partial \Phi}{\partial x} = 0 \text{ (no variation)} \tag{4}$$

Employing equations (3) and (4) in (1) and (2), we obtain

$$v\frac{\partial u}{\partial y} = 2\,\Omega\, v \sin\phi \tag{5}$$

$$v\frac{\partial v}{\partial y} = -2\,\Omega\, u \sin\phi - g\frac{\partial \Phi}{\partial y} \tag{6}$$

If we define $f = 2\,\Omega \sin\phi$ the Coriolis parameter, then equations (5) and (6) can be rewritten in the form

$$v\frac{\partial u}{\partial y} = fv \tag{5a}$$

$$v\frac{\partial v}{\partial y} = -fu - g\frac{\partial \Phi}{\partial y} \tag{6a}$$

There are two solutions to equation (5a). These are :

(i) $v = 0 \quad \frac{\partial u}{\partial y} \neq f$

(ii) $v \neq 0 \quad \frac{\partial u}{\partial y} = f$

Now consider case (i)

$$v = 0 \quad \frac{\partial u}{\partial y} \neq f$$

Substitution into equation (6a) gives

$$u = -\frac{g}{f}\frac{\partial \Phi}{\partial y} \text{ (geostrophic wind)} \tag{7}$$

If $u \approx u_c$ is approximately constant

Then $\frac{\partial \Phi}{\partial y} = -\frac{f u_c}{g}$

If we define $\beta = \frac{\partial f}{\partial y}$ the Rossby parameter (which is assumed constant in the equatorial zone) then $f = \beta y$

$$\frac{\partial \Phi}{\partial y} = -\frac{\beta y u_c}{g}$$

Upon integration we obtain

$$\Phi = \Phi_i - \tfrac{1}{2}\frac{\beta u_c}{g} y^2 \qquad (8)$$

Equation (8) is the same solution given by Johnson and Mörth as the solution pertinent to the 'Bridge' and 'Duct' flow patterns. This solution is of little interest to us in this particular problem.

We now consider case (ii) where solutions to equation (5a) is given by

$$v \neq 0\,; \quad \frac{\partial u}{\partial y} = f$$

Substitution for f in equation (6a) gives

$$v\frac{\partial v}{\partial y} = -u\frac{\partial u}{\partial y} - g\frac{\partial \Phi}{\partial y} \qquad (9)$$

or

$$\frac{\partial}{\partial y}\tfrac{1}{2}(v^2 + u^2) = -g\frac{\partial \Phi}{\partial y}$$

Integration of equation (9) gives

$$u^2 + v^2 = u_i^2 + v_i^2 + 2g(\Phi_i - \Phi) \qquad (10)$$

the subscript i denotes the initial values.

If further we integrate the shear equation

$$\frac{\partial u}{\partial y} = f = \beta y$$

we obtain

$$u = u_i + \tfrac{1}{2}\beta(y^2 - y_i^2) \qquad (11)$$

Equations (10) and (11) are the Gordon, and Johnson and Mörth solutions for the drift flow from one hemisphere to the other down the pressure gradient.

Although the computed trajectories resemble at some latitudes the observed flow pattern in the Indian Ocean SE/SW monsoon flow, a comparison of the observed speeds shows that the equations are unrealistic and therefore should not be used for computations of important derived quantities such as divergence and vorticity.

If we start with the observed wind speed at latitude 10°S and taking the contour gradient poleward of 5 degrees as 9 metres per 2 degrees latitude but equatorward of 5 degrees as 9 metres per 4 degrees of latitude, then the computed wind field from equations (10) and (11) is given below in Table 2. Comparison of Table 2 with Table 1 shows that the computed wind speeds are markedly different from the observed values.

TABLE 2

Latitude	Easterly/Westerly (knots)	Southerly/Northerly (knots)
10°N	20 easterly	71 southerly
5°N	41 easterly	45 southerly
0 (Equator)	48 easterly	24 southerly
5°S	41 easterly	20 southerly
10°S	20 easterly	Nil

The reason for the differences in the tables is that the shear equation $\frac{\partial u}{\partial y} = f$ is unrealistic. This equation is the limiting stable shear on the anticyclonic side of jet streams and would represent a westerly shear of 28 knots from the equator to latitude 10 degrees or 110 knots from the equator to latitude 20 degrees. As is evident in Table 1, the shear between the equator and latitude 10 degrees is roughly true in the hemisphere which is being invaded, but the opposite is the case in the hemisphere from which the drift has come. Further, the shear equation $\frac{\partial u}{\partial y} = f$ holds when a complete ring of air moves north or south, with conservation of angular momentum and this condition is known not to be satisfied by the drift flow.

A MORE REALISTIC MODEL

Table 1 shows that there is symmetry of the flow about the equator, where the zonal component of the wind is zero.

A further observational evidence is that the relative vorticity represented by $\frac{\partial u}{\partial y}$ (westerly shear) is constant with latitude and as may be estimated from Table 1 it is 2 kt/degree latitude between latitudes 10°S and 10°N.

If we use the $x-y$ co-ordinate axis, with the x axis oriented along the equator and the y axis pointing northward, then the equations that approximately fit the observed wind are given by

$$u = 2y \tag{12}$$

$$v = 20 - 0.4y^2 \tag{13}$$

$$v = 0 \text{ at approximately 7 degrees.}$$

Equations (12) and (13) satisfy the observed low level characteristics of the drift flow over the Indian Ocean in July as tabulated in Table 1. Equation (12) shows that with our choice of co-ordinate axis, the winds are easterly in the Southern Hemisphere from where the drift originates, becoming weaker as one approaches the equator, where it becomes zero and thereafter changing to westerlies which becomes stronger as one moves away from the equator. It also shows that within the range of validity of the equation the relative vorticity is constant $\frac{\partial u}{\partial y} = 2$. Equation (13) on the other hand shows that the southerly wind is also symmetric about the equator ($y = 0$) in the latitude zones of interest. The southerly is a maximum at the equator with a magnitude of 20 knots.

One unit of y corresponds to one degree of latitude. This unit of y also satisfies the equation (12) if u is in knots and y is in degrees of latitude. The origin can be placed anywhere at the equator (y=0).

It is desirable to derive the equations for the streamlines, in order to give a complete mathematical description of the wind pattern. The equation of the streamline is given by

$$\frac{v}{u} = \frac{dy}{dx} \qquad (14)$$

Hence $$\frac{20 - 0{\cdot}4y^2}{2y} = \frac{dy}{dx}$$

or $$-\frac{y^2/5 - 10}{y} = \frac{dy}{dx}$$

$$\therefore \qquad \int dx = \int -\frac{5y}{y^2 - 50}\, dy$$

$$\therefore \qquad -(x + C) = \frac{5}{2} \log_e |50 - y^2| \qquad (15)$$

or $$e^{-0{\cdot}4(x+C)} = 50 - y^2$$

The equation (15) is valid provided $50 > y^2$. This limits the latitude range of validity to about 7°N to 7°S. C is the constant of integration and hence determines the family of the streamlines. A particular value of C will give the equation of a streamline pattern.

Table 3 shows the values of x for various values of y as C takes on different values.

TABLE 3

y	Values of x		
	$C = 0$	$-C = 10$	$-C = 15$
0	—9.7825	0.2175	5.2175
1	—9.7300	0.2700	5.2700
2	—9.5700	0.4300	5.4300
3	—9.2825	0.7175	5.7175
4	—8.8200	1.1800	6.1800
5	—8.0500	1.9500	6.9500
6	—6.5950	3.4050	8.4050
7	0	10	15

The values reported in Table 3 above are plotted in Fig. 7. Note that the mathematical constraint is such that the pattern is only valid between 7°S and 7°N. The streamlines suggest the existence of an asymptote of divergence poleward of latitude 7°S and a corresponding asymptote of convergence poleward of latitude 7°N.

Vorticity and Divergence :

The relative vorticity of the flow is given by

$$\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\approx-\frac{\partial u}{\partial y}=-2 \text{ knots/degree}$$

with our assumption that there is no longitudinal variation of the field. The divergence is given by

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\approx\frac{\partial v}{\partial y}=-0{\cdot}8y$$

This divergence is positive at all latitudes south of the equator and changing into convergence north of the equator. The magnitudes expressed as a function of latitude are shown in table 4. The profile of the divergence field is shown in figure 8.

TABLE 4. *Magnitude of the Divergence values at various latitude circles. Units are in knots/degree.*

Latitude	Divergence
0	0
1	0.8
2	1.6
3	2.4
4	3.2
5	4.0
6	4.8
7	5.6

Note that the divergence pattern as shown in Fig. 8 bears a remarkable agreement with the results obtained by Asnani and Pisharoty (1965) who employed theoretical consideration for inferring the divergence field in the southeast/southwest monsoon.

It often happens that when qualitative estimates of the divergence field is made in the drift flow, the relative vorticity is assumed to be due mainly to the curvature term when the latter is expressed in natural co-ordinates. For this reason it is of interest to compute the radius of curvature from the streamline equation.

The radius of curvature K is given by the expression

$$K=\frac{\dfrac{d^2y}{dx^2}}{\left[1+\left(\dfrac{dy}{dx}\right)^2\right]^{\frac{3}{2}}} \tag{16}$$

When this expression is evaluated using equation (15), one obtains the values as tabulated in the table 5 below. The units are in per degree of latitude and the values of K are negative north of the equator and positive south of the equator.

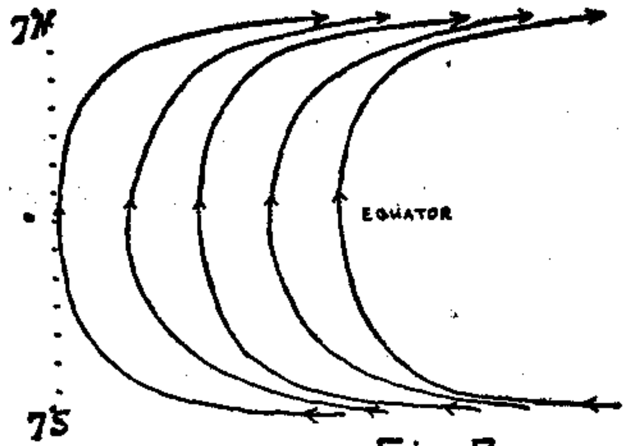

Fig.7

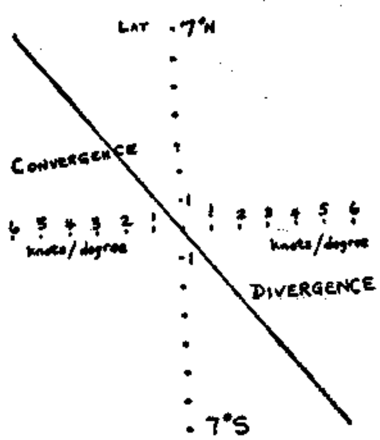

Fig.8

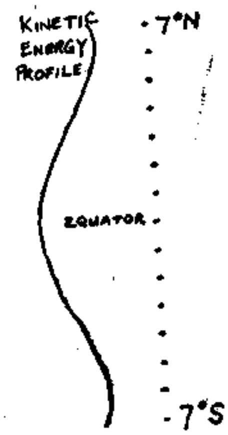

Fig.10

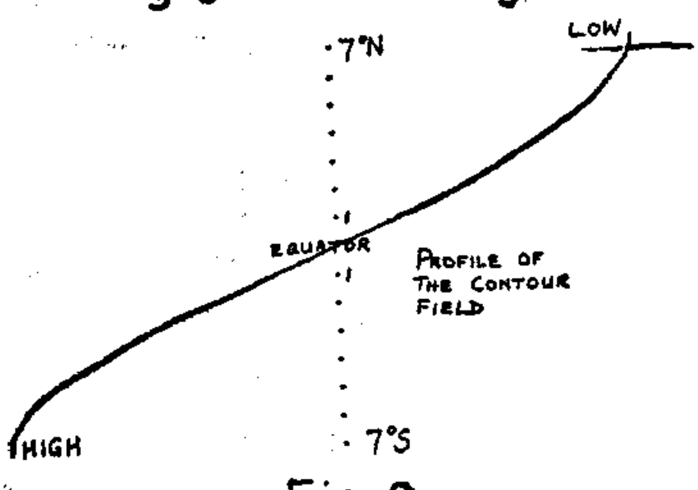

Fig.9

Fig. 7. Streamline pattern obtained from the equation of the streamline field that fits the empirical data of the low level Indian Ocean SE/SW monsoon in July.

Fig. 8. The divergence pattern of the low level SE/SW monsoon over the Indian Ocean in July.

Fig. 9. The profile of the contour field of the SE/SW monsoon.

Fig. 10. Distribution of kinetic energy of the low level SE/SW monsoon.

TABLE 5

Latitude	K	K'
0	—	.10
1	.10	
2	.12	
3	.15	
4	.18	
5	.12	.20
6	.17	
7	.01	
10	—	.10

The column marked K' is the curvature as computed from table 1 and the observed property that the negative of the relative vorticity in this drift flow is 2 knots/degree.

THE CONTOUR FIELD

As may be inferred from Fig. 6a, the meridional profile of the contour field should be as shown in Fig. 9. It is not consistent to derive the profile from equation (9) as this will imply that the shear equation $\frac{\partial u}{\partial y} = f$ is valid for the monsoon flow. The profile as shown in Fig. 9 is, however, consistent with the observed zonal wind distribution with easterlies south of the equator and westerlies north of the equator. If for a moment we consider only the zonal component of motion, then equation (2) may be written as

$$\beta y u = -g \frac{\partial \bar{\phi}}{\partial y} \tag{17}$$

$$\text{using} \qquad u = 2y \tag{18}$$

and upon integration one obtains

$$\tfrac{2}{3}\beta y^3 + g\bar{\phi} = \tfrac{2}{3}\beta y_i^3 + g\bar{\phi}_i \tag{19}$$

showing that $\bar{\phi}$ decreases from 7°S to the lowest value at 7°N.

KINETIC ENERGY

The distribution of the kinetic energy is proportional to

$$200 - 6y^2 + 0.08y^4.$$

This value is plotted in figure 10. As to be expected in the observed windfield the maximum is at the equator with a minimum centred between latitudes 6°N and 7°N and another between 6°S and 7°S.

Other quantities such as the distribution of momentum and the transport of both momentum and kinetic energy may easily be derived for this flow.

Generalisation of the Present Treatment

The equations (12) and (13) are special cases whose constants have been fixed in accordance with the July SE/SW monsoon flow over Indian Ocean. A general expression for the drift flow may be expressed in the form

$$u = ay \tag{20}$$

$$v = b + cy - dy^2 \tag{21}$$

where a, b, c, and d are positive constants.

The equations for the streamlines are given by

$$\frac{v}{u} = \frac{dy}{dx}$$

When equations (20) and (21) are substituted into the above expressions we obtain the general solutions as

$$x + B = -\frac{a}{2d}\log_e|dy^2 - cy - b| + \frac{ac}{a\sqrt{c^2+4db}}\tanh^{-1}\frac{2dy-c}{\sqrt{c^2+4db}}$$

$$\text{if } (2dy-c)^2 < c^2 + 4db$$

$$= -\frac{a}{2d}\log_e|dy^2 - cy - b| + \frac{ac}{d\sqrt{c^2+4db}}\operatorname{ctnh}^{-1}\frac{2dy-c}{\sqrt{c^2+4db}}$$

$$\text{if } (2dy-c)^2 > c^2 + 4db$$

$$= -\frac{a}{2d}\log_e|dy^2 - cy - b| + \frac{ac}{d(2dy-c)}$$

$$\text{if } c^2 = 4db$$

This generalised solution permits the divergence to be positive north of the equator. This property is in accord with observations in most of the drift flows that are observed over tropical regions.

Conclusion

This paper is a preliminary attempt at the description of a circulation pattern that is of common features over the Indian Ocean during July/August. For simplicity the problem which is a three dimensional one has been treated in two dimensions. There is little doubt that to completely understand the SE/SW monsoon flow the transport and balance of humidity, temperature and energy has to be incorporated in future treatments.

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